Seventh Internatioaal Olympiad, 1965

1965/1.

Determine all values x in the interval $0 \leq x \leq 2\pi$ which satisfy the inequality

$$2\cos x \leq \left|\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}\right| \leq \sqrt{2}.$$

1965/2.

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

with unknowns x_1, x_2, x_3. The coefficients satisfy the conditions:
(a) a_{11}, a_{22}, a_{33} are positive numbers;
(b) the remaining coefficients are negative numbers;
(c) in each equation, the sum of the coefficients is positive.
Prove that the given system has only the solution $x_1 = x_2 = x_3 = 0$.

1965/3.

Given the tetrahedron $ABCD$ whose edges AB and CD have lengths a and b respectively. The distance between the skew lines AB and CD is d, and the angle between them is ω. Tetrahedron $ABCD$ is divided into two solids by plane ε, parallel to lines AB and CD. The ratio of the distances of ε from AB and CD is equal to k. Compute the ratio of the volumes of the two solids obtained.

1965/4.

Find all sets of four real numbers x_1, x_2, x_3, x_4 such that the sum of any one and the product of the other three is equal to 2.

1965/5.

Consider ΔOAB with acute angle AOB. Through a point $M \neq O$ perpendiculars are drawn to OA and OB, the feet of which are P and Q respectively. The point of intersection of the altitudes of ΔOPQ is H. What is the locus of H if M is permitted to range over (a) the side AB, (b) the interior of ΔOAB?

1965/6.

In a plane a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n.