Ninth International Olympiad, 1967

1967/1.

Let $ABCD$ be a parallelogram with side lengths $AB = a, AD = 1$, and with $\angle BAD = \alpha$. If ΔABD is acute, prove that the four circles of radius 1 with centers A, B, C, D cover the parallelogram if and only if

$$a \leq \cos\alpha + \sqrt{3}\sin\alpha.$$

1967/2.

Prove that if one and only one edge of a tetrahedron is greater than 1, then its volume is $\leq 1/8$.

1967/3.

Let k, m, n be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s+1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \cdots (c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \cdots c_n$.

1967/4.

Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangles ABC that are similar to $\Delta A_1B_1C_1$ (so that vertices A_1, B_1, C_1 correspond to vertices A, B, C, respectively) and circumscribed about triangle $A_0B_0C_0$ (where A_0 lies on BC, B_0 on CA, and AC_0 on AB). Of all such possible triangles, determine the one with maximum area, and construct it.

1967/5.

Consider the sequence $\{c_n\}$, where

$$\begin{array}{rcl} c_1 & = & a_1 + a_2 + \cdots + a_8 \\ c_2 & = & a_1^2 + a_2^2 + \cdots + a_8^2 \\ & & \cdots \\ c_n & = & a_1^n + a_2^n + \cdots + a_8^n \\ & & \cdots \end{array}$$

in which $a_1, a_2, \cdots, a_8$ are real numbers not all equal to zero. Suppose that an infinite number of terms of the sequence $\{c_n\}$ are equal to zero. Find all natural numbers n for which $c_n = 0$.

1967/6.

In a sports contest, there were m medals awarded on n successive days ($n > 1$). On the first day, one medal and 1/7 of the remaining $m - 1$ medals were awarded. On the second day, two medals and 1/7 of the now remaining medals were awarded; and so on. On the n-th and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether?