Tenth International Olympiad, 1968

1968/1.

Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

1968/2.

Find all natural numbers x such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

1968/3.

Consider the system of equations

$$\begin{aligned} ax_1^2 + bx_1 + c &= x_2 \\ ax_2^2 + bx_2 + c &= x_3 \\ &\cdots \\ ax_{n-1}^2 + bx_{n-1} + c &= x_n \\ ax_n^2 + bx_n + c &= x_1, \end{aligned}$$

with unknowns $x_1, x_2, \cdots, x_n$, where a, b, c are real and $a \neq 0$. Let $\Delta = (b-1)^2 - 4ac$. Prove that for this system
(a) if $\Delta < 0$, there is no solution,
(b) if $\Delta = 0$, there is exactly one solution,
(c) if $\Delta > 0$, there is more than one solution.

1968/4.

Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

1968/5.

Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

holds for all x.
(a) Prove that the function f is periodic (i.e., there exists a positive number b such that $f(x+b) = f(x)$ for all x).
(b) For $a = 1$, give an example of a non-constant function with the required properties.

1968/6.

For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right]=\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\cdots+\left[\frac{n+2^k}{2^{k+1}}\right]+\cdots$$

(The symbol $[x]$ denotes the greatest integer not exceeding x.)