

Seventeenth International Olympiad, 1975

1975/1.

Let x_i, y_i ($i = 1, 2, \dots, n$) be real numbers such that

$$x_1 \geq x_2 \geq \dots \geq x_n \text{ and } y_1 \geq y_2 \geq \dots \geq y_n.$$

Prove that, if z_1, z_2, \dots, z_n is any permutation of y_1, y_2, \dots, y_n , then

$$\sum_{i=1}^n (x_i - y_i)^2 \leq \sum_{i=1}^n (x_i - z_i)^2.$$

1975/2.

Let a_1, a_2, a_3, \dots be an infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many a_m which can be written in the form

$$a_m = xa_p + ya_q$$

with x, y positive integers and $q > p$.

1975/3.

On the sides of an arbitrary triangle ABC , triangles ABR, BCP, CAQ are constructed externally with $\angle CBP = \angle CAQ = 45^\circ, \angle BCP = \angle ACQ = 30^\circ, \angle ABR = \angle BAR = 15^\circ$. Prove that $\angle QRP = 90^\circ$ and $QR = RP$.

1975/4.

When 4444^{4444} is written in decimal notation, the sum of its digits is A . Let B be the sum of the digits of A . Find the sum of the digits of B . (A and B are written in decimal notation.)

1975/5.

Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.

1975/6.

Find all polynomials P , in two variables, with the following properties:

(i) for a positive integer n and all real t, x, y

$$P(tx, ty) = t^n P(x, y)$$

(that is, P is homogeneous of degree n),

(ii) for all real a, b, c ,

$$P(b + c, a) + P(c + a, b) + P(a + b, c) = 0,$$

(iii) $P(1, 0) = 1$.