27$^{\text{th}}$ International Mathematical Olympiad

Warsaw, Poland

Day I

July 9, 1986

1. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

2. A triangle $A_1 A_2 A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a set of points P_1, P_2, P_3, ..., such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle $120°$ clockwise (for $k = 0$, 1, 2, ...). Prove that if $P_{1986} = P_0$, then the triangle $A_1 A_2 A_3$ is equilateral.

3. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to and end after a finite number of steps.

4. Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) in the plane having center at O. A triangle XYZ, which is congruent to and initially conincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.

5. Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:

   (i) $f(xf(y))f(y) = f(x + y)$ for all x, $y \geq 0$,

   (ii) $f(2) = 0$,

   (iii) $f(x) \neq 0$ for $0 \leq x < 2$.

6. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1?