29^{th} International Mathematical Olympiad

Canberra, Australia

Day I

1. Consider two coplanar circles of radii R and r $(R > r)$ with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then $A = P$.)

   (i) Find the set of values of $BC^2 + CA^2 + AB^2$.

   (ii) Find the locus of the midpoint of BC.

2. Let n be a positive integer and let $A_1, A_2, \ldots, A_{2n+1}$ be subsets of a set B. Suppose that

   (a) Each A_i has exactly $2n$ elements,

   (b) Each $A_i \cap A_j$ $(1 \le i < j \le 2n+1)$ contains exactly one element, and

   (c) Every element of B belongs to at least two of the A_i.

   For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements?

3. A function f is defined on the positive integers by

$$\begin{aligned} f(1) &= 1, \quad f(3) = 3, \\ f(2n) &= f(n), \\ f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n), \end{aligned}$$

   for all positive integers n.

   Determine the number of positive integers n, less than or equal to 1988, for which $f(n) = n$.

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Day II

4. Show that set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

5. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$.

6. Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.