

**33<sup>rd</sup> International Mathematical Olympiad**

**First Day - Moscow - July 15, 1992**

**Time Limit: 4½ hours**

1. Find all integers  $a, b, c$  with  $1 < a < b < c$  such that

$$(a-1)(b-1)(c-1) \quad \text{is a divisor of } abc - 1.$$

2. Let  $\mathbf{R}$  denote the set of all real numbers. Find all functions  $f : \mathbf{R} \rightarrow \mathbf{R}$  such that

$$f(x^2 + f(y)) = y + (f(x))^2 \quad \text{for all } x, y \in \mathbf{R}.$$

3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of  $n$  such that whenever exactly  $n$  edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

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1. In the plane let  $C$  be a circle,  $L$  a line tangent to the circle  $C$ , and  $M$  a point on  $L$ . Find the locus of all points  $P$  with the following property: there exists two points  $Q, R$  on  $L$  such that  $M$  is the midpoint of  $QR$  and  $C$  is the inscribed circle of triangle  $PQR$ .
2. Let  $S$  be a finite set of points in three-dimensional space. Let  $S_x, S_y, S_z$  be the sets consisting of the orthogonal projections of the points of  $S$  onto the  $yz$ -plane,  $zx$ -plane,  $xy$ -plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where  $|A|$  denotes the number of elements in the finite set  $A$ . (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

3. For each positive integer  $n$ ,  $S(n)$  is defined to be the greatest integer such that, for every positive integer  $k \leq S(n)$ ,  $n^2$  can be written as the sum of  $k$  positive squares.
- (a) Prove that  $S(n) \leq n^2 - 14$  for each  $n \geq 4$ .
  - (b) Find an integer  $n$  such that  $S(n) = n^2 - 14$ .
  - (c) Prove that there are infinitely many integers  $n$  such that  $S(n) = n^2 - 14$ .