43rd IMO 2002

Problem 1. S is the set of all (h, k) with h, k non-negative integers such that $h + k < n$. Each element of S is colored red or blue, so that if (h, k) is red and $h' \leq h, k' \leq k$, then (h', k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

Problem 2. BC is a diameter of a circle center O. A is any point on the circle with $\angle AOC > 60^o$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.

Problem 3. Find all pairs of integers $m > 2, n > 2$ such that there are infinitely many positive integers k for which $k^n + k^2 - 1$ divides $k^m + k - 1$.

Problem 4. The positive divisors of the integer $n > 1$ are $d_1 < d_2 < \ldots < d_k$, so that $d_1 = 1, d_k = n$. Let $d = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

Problem 5. Find all real-valued functions on the reals such that $(f(x) + f(y))((f(u) + f(v)) = f(xu - yv) + f(xv + yu)$ for all x, y, u, v.

Problem 6. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, \cdots, O_n$. Show that $\sum_{i<j} 1/O_i O_j \leq (n - 1)\pi/4$.