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Problem 1. Six points are chosen on the sides of an equilateral triangle ABC : A_1, A_2 on BC , B_1, B_2 on CA and C_1, C_2 on AB , such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 2. Let a_1, a_2, \dots be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers a_1, a_2, \dots, a_n leave n different remainders upon division by n . Prove that every integer occurs exactly once in the sequence a_1, a_2, \dots .

Problem 3. Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Problem 4. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \quad n \geq 1.$$

Problem 5. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA . Let two variable points E and F lie of the sides BC and DA , respectively and satisfy $BE = DF$. The lines AC and BD meet at P , the lines BD and EF meet at Q , the lines EF and AC meet at R .

Prove that the circumcircles of the triangles PQR , as E and F vary, have a common point other than P .

Problem 6. In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.