

49th INTERNATIONAL MATHEMATICAL OLYMPIAD
MADRID (SPAIN), JULY 10-22, 2008

Wednesday, July 16, 2008

Problem 1. An acute-angled triangle ABC has orthocentre H . The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2 . Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at B_1 and B_2 , and the circle passing through H with centre the midpoint of AB intersects the line AB at C_1 and C_2 . Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

Problem 2. (a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers x, y, z , each different from 1, and satisfying $xyz = 1$.

(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z , each different from 1, and satisfying $xyz = 1$.

Problem 3. Prove that there exist infinitely many positive integers n such that $n^2 + 1$ has a prime divisor which is greater than $2n + \sqrt{2n}$.

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Problem 4. Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z , satisfying $wx = yz$.

Problem 5. Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled $1, 2, \dots, 2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequences of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off.

Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine the ratio N/M .

Problem 6. Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C , which is also tangent to the lines AD and CD . Prove that the common external tangents of ω_1 and ω_2 intersect on ω .