



Tuesday, July 10, 2012

**Problem 1.** Given triangle  $ABC$  the point  $J$  is the centre of the excircle opposite the vertex  $A$ . This excircle is tangent to the side  $BC$  at  $M$ , and to the lines  $AB$  and  $AC$  at  $K$  and  $L$ , respectively. The lines  $LM$  and  $BJ$  meet at  $F$ , and the lines  $KM$  and  $CJ$  meet at  $G$ . Let  $S$  be the point of intersection of the lines  $AF$  and  $BC$ , and let  $T$  be the point of intersection of the lines  $AG$  and  $BC$ . Prove that  $M$  is the midpoint of  $ST$ .

(The *excircle* of  $ABC$  opposite the vertex  $A$  is the circle that is tangent to the line segment  $BC$ , to the ray  $AB$  beyond  $B$ , and to the ray  $AC$  beyond  $C$ .)

**Problem 2.** Let  $n \geq 3$  be an integer, and let  $a_2, a_3, \dots, a_n$  be positive real numbers such that  $a_2 a_3 \cdots a_n = 1$ . Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

**Problem 3.** The *liar's guessing game* is a game played between two players  $A$  and  $B$ . The rules of the game depend on two positive integers  $k$  and  $n$  which are known to both players.

At the start of the game  $A$  chooses integers  $x$  and  $N$  with  $1 \leq x \leq N$ . Player  $A$  keeps  $x$  secret, and truthfully tells  $N$  to player  $B$ . Player  $B$  now tries to obtain information about  $x$  by asking player  $A$  questions as follows: each question consists of  $B$  specifying an arbitrary set  $S$  of positive integers (possibly one specified in some previous question), and asking  $A$  whether  $x$  belongs to  $S$ . Player  $B$  may ask as many such questions as he wishes. After each question, player  $A$  must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any  $k + 1$  consecutive answers, at least one answer must be truthful.

After  $B$  has asked as many questions as he wants, he must specify a set  $X$  of at most  $n$  positive integers. If  $x$  belongs to  $X$ , then  $B$  wins; otherwise, he loses. Prove that:

1. If  $n \geq 2^k$ , then  $B$  can guarantee a win.
2. For all sufficiently large  $k$ , there exists an integer  $n \geq 1.99^k$  such that  $B$  cannot guarantee a win.



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MAR DEL PLATA - ARGENTINA

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Day: 2

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**Problem 4.** Find all functions  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  such that, for all integers  $a, b, c$  that satisfy  $a + b + c = 0$ , the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here  $\mathbb{Z}$  denotes the set of integers.)

**Problem 5.** Let  $ABC$  be a triangle with  $\angle BCA = 90^\circ$ , and let  $D$  be the foot of the altitude from  $C$ . Let  $X$  be a point in the interior of the segment  $CD$ . Let  $K$  be the point on the segment  $AX$  such that  $BK = BC$ . Similarly, let  $L$  be the point on the segment  $BX$  such that  $AL = AC$ . Let  $M$  be the point of intersection of  $AL$  and  $BK$ .

Show that  $MK = ML$ .

**Problem 6.** Find all positive integers  $n$  for which there exist non-negative integers  $a_1, a_2, \dots, a_n$  such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \dots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \dots + \frac{n}{3^{a_n}} = 1.$$

Language: English

Time: 4 hours and 30 minutes  
Each problem is worth 7 points