Saturday, 8. July 2023

Problem 1. Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leqslant i \leqslant k-2$.

Problem 2. Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$.
Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Problem 3. For each integer $k \geqslant 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$, where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \geqslant 1$.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

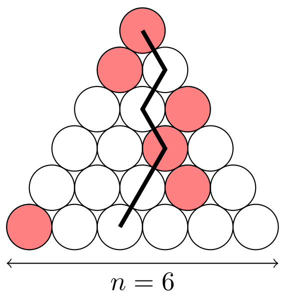
Sunday, 9. July 2023

Problem 4. Let $x_1, x_2, \dots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \dots, 2023$. Prove that $a_{2023} \geqslant 3034$.

Problem 5. Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \dots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A *ninja path* in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Problem 6. Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Let BC_1 and CB_1 meet at A_2, let CA_1 and AC_1 meet at B_2, and let AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, then the three circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points.

(Note: a scalene triangle is one where no two sides have equal length.)